Application of a Variation–Iteration Method to Waveguides with Inhomogeneous Lossy Loads

AUGUSTE A. LALOUX, STUDENT MEMBER, IEEE, RENÉ J. M. GOVAERTS, MEMBER, IEEE, AND ANDRÉ S. VANDER VORST, SENIOR MEMBER, IEEE

Abstract—**An approximate technique for solving eigenvalue equations, the variation–iteration method, is commonly used in theoretical physics. A previous paper presented the application of this method to the scalar case of a dielectric slab loaded rectangular waveguide. This paper presents its extension to the complex vector case of a lossy dielectric insert loaded waveguide.**

Starting from an initial trial function, iterates are calculated in which the components relative to the unwanted eigenfunctions are eliminated. Both an upper and a lower bound for the unknown eigenvalues are available. Each iterate is the solution of a system of algebraic simultaneous equations. This system is solved by the successive overrelaxation method using an automatically computed optimal accelerating factor. An extrapolation technique further accelerates the convergence.

This yields the attenuation and propagation coefficients for the dominant as well as several other modes, together with the electric and magnetic field configurations.

I. Introduction

IN A PREVIOUS PAPER [1], a computerized version of the variation–iteration method, previously used in theoretical physics was proposed. To check the validity of the method, it was applied to the well-known problem of the dielectric slab loaded waveguide. It was shown that the variation–iteration technique offers many advantages with respect to other procedures. It produces both an upper and lower bound for the approximation of the obtained solution, with respect to the exact unknown solution. An extrapolation technique may be used to accelerate the convergence.

Rayleigh–Ritz and Galerkin procedures for waveguides loaded with dielectric inserts have been used [2]–[6] but the convergence of moment methods has been shown to be a rather slow one [7] and alternate approaches are needed (e.g., the modal technique [8]).

The purpose of the present paper is to extend the variation–iteration procedure to vector problems (dielectric inserts which do not extend across either of the waveguide transverse dimensions) and complex problems (lossy loads).

In the first part, the basic outline of the variation–iteration method is given. In the second part, the numerical treatment is developed for the lossless case while the third part is devoted to the lossy case.

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A. A. Laloux and A. S. Vander Vorst are with the Microwave Laboratory, Catholic University of Louvain, Louvain-la-Neuve, Belgium.

R. J. M. Govaerts is with the Circuits and Systems Division, Catholic University of Louvain, Heverlee, Belgium.

II. Basic Outline of the Variation–Iteration Procedure

The variation–iteration procedure for solving eigenvalue equations is basically a particular case of the Wielandt iteration method [9], [10] consisting of an iterative process combined with a variational principle. This procedure results in a method for systematically improving upon a trial function, which converges towards the eigenfunction of the problem while the eigenvalue is refined by a Rayleigh quotient. It produces both an upper and lower bound for the eigenvalue. Only the basic outline of the method will be given here; the reader interested in having more details is referred to [11].

Using a formal operator notation, the eigenvalue problem is characterized by

$$\mathfrak{L}f = \lambda \mathfrak{M} f. \tag{1}$$

The sequence of unknown eigenvalues $\lambda_p (\lambda_0 < \lambda_1 < \lambda_2 < \cdots)$ corresponds to the eigenfunctions X_p.

In this presentation, the eigenfunctions are supposed to be real and scalar; the extension to the complex vector case is straightforward.

Let f_0 be an initial trial function. We define the nth iterate by

$$f_n = \mathfrak{L}^{-1} \mathfrak{M} f_{n-1}. \tag{2}$$

Notice that the factor λ is dropped; this means that instead of solving an eigenvalue problem, one only calculates a function from another one (for each iterate). This is the main advantage of the variation–iteration technique which avoids one of the inherent difficulties of an eigenvalue problem.

Assuming that the eigenfunctions X_p of the problem form a complete set (this will always be the case if the problem is self-adjoint or if the eigenvalues are real and distinct [12]), the trial function may be expanded:

$$f_0 = \sum c_p X_p. \tag{3}$$

Hence, from (1)–(3),

$$f_n = \sum \frac{c_p}{\lambda_p^n} X_p \tag{4}$$

which shows that, in the absence of a degeneracy (i.e., $\lambda_0 = \lambda_1$), the set f_n converges to the eigenfunction corresponding to the lowest eigenvalue X_0 by elimination of the unwanted components contained in the trial function (in the complex case, the moduli have to be taken into

account). The greater the ratio λ_1/λ_0, the faster the convergence.

Defining the adjoint problem of (1) by

$$\mathfrak{L}^a g^a = \lambda \mathfrak{M}^a g^a \tag{5}$$

where $\mathfrak{L}^a$ and $\mathfrak{M}^a$ are, respectively, the adjoints of $\mathfrak{L}$ and $\mathfrak{M}$, and denoting by (f_1, f_2) the scalar product of f_1 and f_2, it can be shown that

$$\lambda = \frac{(\mathfrak{L}f, g^a)}{(\mathfrak{M}f, g^a)} \tag{6}$$

is a variational expression for the eigenvalues as a function of the eigenfunctions, presenting a stationary value when f and g^a are, respectively, the eigenfunctions of (1) and (5). The various iterates may be inserted into (6). This gives, after some rewriting, the following approximations to λ_0 (known as Rayleigh quotients):

$$\lambda_0^{(n)} = \frac{(\mathfrak{L}f_n, g_n^a)}{(\mathfrak{M}f_n, g_n^a)} = \frac{(\mathfrak{M}f_{n-1}, g_n^a)}{(\mathfrak{M}f_n, g_n^a)} \tag{7}$$

$$\lambda_0^{(n+1/2)} = \frac{(\mathfrak{M}f_n, g_n^a)}{(\mathfrak{M}\mathfrak{L}^{-1}\mathfrak{M}f_n, g_n^a)} = \frac{(\mathfrak{M}f_n, g_n^a)}{(\mathfrak{M}f_{n+1}, g_n^a)}. \tag{8}$$

When the operators are real and positive-definite, it may be shown that the set $\lambda_0^{(a)}$ including both integral and half-integral values of a forms a monotonic sequence of decreasing values, approaching the exact value from above if the products $c_p c_p'$ all have the same sign (the c_p are the coefficients of expansion (3), the c_p' are the coefficients appearing in the expansion of g_0^a).

If the problem is self-adjoint, $c_p = c_p'$ and the monotonic convergence is obtained. If the problem is not self-adjoint, the signs of the c_p and c_p' depend upon the (arbitrary) choice of the trial functions. Hence it may happen that the sequence $\lambda_0^{(a)}$ is not a monotonic one. However, while performing the iterations, the components of f_n relative to the unwanted eigenfunctions are progressively eliminated. When the iterations have proceeded so far that f_n is but a mixture of X_0 and X_1, a monotonic (increasing or decreasing) convergence of the $\lambda_0^{(a)}$ towards λ_0 does appear. This emphasizes the importance of a good trial function. In the complex case, there is no proof that the convergence is monotonic, but in all the experimental results, it appears to be so.

If the ratio λ_1/λ_0 is close to unity, convergence may become rather slow. An extrapolation is available [11] for this case which permits the unknown eigenvalue to be evaluated in spite of the slow convergence. Of course, it may also be used whenever the convergence is satisfactory, but it is usually not necessary.

An estimate for λ_1 may be used in the computation of a lower bound for λ_0. If three successive iterates are calculated and if the iterations have proceeded so far that one has

$$\lambda_0^{(n+1)}\lambda_0^{(n+3/2)} < \lambda_1^2 \tag{9}$$

then

$$\lambda_0 > \lambda_0^{(n+3/2)}\left(1 - \frac{\lambda_0^{(n+1)}(\lambda_0^{(n+1/2)} - \lambda_0^{(n+3/2)})}{\lambda_1^2 - \lambda_0^{(n+3/2)}\lambda_0^{(n+1)}}\right). \tag{10}$$

The higher order eigenvalues are evaluated by the same procedure. They may be obtained either by starting from trial functions orthogonal with all the lower order eigenfunctions or by orthogonalizing after each iteration with all these eigenfunctions (Gram–Schmidt procedure). Defining an orthogonalizing function, linear in X_i and X_j so that

$$F(X_i, X_j) = 0, \qquad \text{for all } i \neq j \tag{11}$$

it appears that

$$F(X_0, f_0) = c_0 F(X_0, X_0). \tag{12}$$

The c_0X_0 component may then be removed:

$$(f_0)\perp_0 = f_0 - X_0\frac{F(X_0, f_0)}{F(X_0, X_0)}. \tag{13}$$

Applying the variation–iteration method to $(f_0)\perp_0$ leads to the lowest order eigenfunction present in $(f_0)\perp_0, X_1$.

III. Application to a Waveguide Containing a Dielectric Lossless Insert (Fig. 1)

It appears from Maxwell's equations that the magnetic field $\bar{H}$ may be expressed as the curl of a vector potential $\bar{P}$

$$\bar{H} = j\omega\epsilon_0 \nabla \times \bar{P} \tag{14}$$

which gives for the electric field

$$\bar{E} = k_0^2 \bar{P} + \nabla S \tag{15}$$

where S is an arbitrary scalar function and $k_0^2 = \omega^2\mu_0\epsilon_0$ is the wavenumber in vacuum. Using Maxwell's equations and specifying the divergence of $\bar{P}$ yields the wave equation

$$\nabla^2\bar{P} - \left(\frac{1}{\epsilon_r}\nabla\epsilon_r\right)\nabla\cdot\bar{P} + \epsilon_r k_0^2 \bar{P} = 0. \tag{16}$$

In a similar way, $\bar{E}$ may be expressed as the curl of a vector potential. It may be shown [13] that both formu-

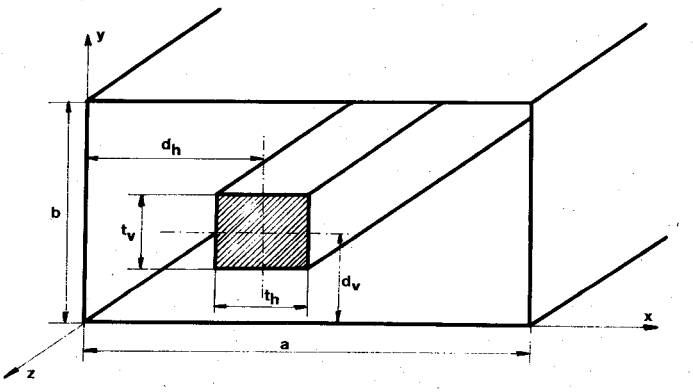

Fig. 1. Rectangular waveguide loaded with a dielectric insert.

lations lead to identical solutions, the first one being presented here because it is better suited to the case of a dielectric load.

In the case of dielectric inserts with boundaries parallel to the waveguide walls, the gradient $\nabla\epsilon_r$ is no longer defined. This can be taken care of by using (16) with $\nabla\epsilon_r = 0$ for the homogeneous regions and introducing interface conditions at the boundaries of these regions.

We now consider fields propagating along the z axis. We can describe them by a potential $\bar{P}$ of the form [14]

$$\bar{P} = (\bar{a}_x P_x(x,y) + \bar{a}_y P_y(x,y))e^{-\gamma z} \tag{17}$$

which separates the vector Helmholtz equation (16) into two scalar equations

$$\nabla_t^2 P_{x,y} + (\epsilon_r k_0^2 + \gamma^2) P_{x,y} = 0. \tag{18}$$

At the waveguide walls, the vanishing of the tangential components of the electric field written as a function of P_x and P_y yields the following conditions:

$$\begin{aligned} &\text{horizontal metallic wall: } P_x = 0 \qquad \partial P_y/\partial y = 0 \\ &\text{vertical metallic wall: } \quad P_y = 0 \qquad \partial P_x/\partial x = 0. \end{aligned} \tag{19}$$

At the dielectric interface, the continuity of the four tangential field components implies the continuity of

$$P_x;\ P_y;\ \left(\frac{\partial P_x}{\partial y} + \frac{\partial P_y}{\partial x}\right);\ \frac{1}{\epsilon_r}\left(\frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y}\right). \tag{20}$$

A linear operator of the second order such as (16) is always formally self-adjoint if the appropriate scalar product is used. But the way the waveguide is divided into homogeneous regions implies the interface conditions (20), the last one being not self-adjoint. This fact will be taken into account by using the generalized Rayleigh quotient. The choice of $\bar{P}$ is, however, very helpful: it yields boundary and interface conditions which do not contain the eigenvalue. This is required for the use of the variation–iteration method. This is not the case when using, for instance, axially directed electric and magnetic vector potentials.

Equation (18) has to be written in the form

$$\mathfrak{L}\bar{P} = \lambda\mathfrak{M}\bar{P}. \tag{21}$$

Among the various possible choices of $\mathfrak{L}$, $\mathfrak{M}$, and λ, it seems more attractive to choose k_0^2 as the eigenvalue because all eigenvalues will be positive for a given γ^2, while for a given frequency, some modes may be evanescent ($\gamma^2 > 0$) and the other ones propagating ($\gamma^2 < 0$). Hence an adequate form for (21) is

$$(\nabla_t^2 + \gamma^2)P = -\epsilon_r k_0^2 P \tag{22}$$

in a homogeneous region, where P stands for P_x or P_y. The propagation constant γ is treated as a parameter; for each value of γ, the corresponding wave number k_0^2 is calculated, yielding the dispersion characteristic of the waveguide.

The iteration equation (2) when applied to (22) requires the solution of a system of two partial differential equations with proper boundary and interface conditions. As the problems we wish to solve are rather complicated (bidimensional, vector, and complex with many interfaces), analytical methods do not appear to be the appropriate ones. Considering the isomorphism existing between operators and their discretized representation as matrices [16] as well as the geometry of the problem, one is led to solve the whole problem by discretization. This will be done according to the finite-difference procedure with a square mesh and the classical five-point formula for the representation of the Laplacian operator. The treatment of the homogeneous Dirichlet and Neumann conditions (19) on the metal boundaries is straightforward. The interface conditions, however, are more difficult to deal with. A horizontal dielectric interface for instance, implies the continuity conditions (20). The first two conditions are realized (in the finite-difference sense) by considering only one value of P_x and P_y for each point on the interface. Writing the other two continuity conditions yields the equations to be verified by P_x and P_y. As an example, the continuity of the fourth expression of (20) will be treated here. With the notations of Fig. 2, one has

$$\begin{aligned} &\frac{1}{\epsilon_N}(P_{xE} - P_{xW} + P_{yN} - P_{yS}(N)) \\ &\qquad = \frac{1}{\epsilon_S}(P_{xE} - P_{xW} + P_{yN}(S) - P_{yS}) \end{aligned} \tag{23}$$

where the notation $P_{yS}(N)$ shows that this point has to be considered as belonging to medium N as if this medium were extending outside of the interface. The unknown values $P_{yS}(N)$ and $P_{yN}(S)$ are eliminated by writing the limiting forms of (22) when the interface is approached from both sides. Hence the iteration relation (2) takes the form

$$\begin{aligned} &\left\{(P_{yE} + P_{yW})\epsilon_m + \epsilon_N P_{yS} + \epsilon_S P_{yN} - (P_{xE} - P_{xW})\right. \\ &\left.\cdot\left(\frac{\epsilon_N - \epsilon_S}{2}\right) - \epsilon_m(4 - \gamma^2 h^2)P_{yP}\right\}_n = -h^2\epsilon_N\epsilon_S(P_{yP})_{n-1} \end{aligned} \tag{24}$$

where $\epsilon_m = \frac{1}{2}(\epsilon_N + \epsilon_S)$. This condition introduces a coupling between P_x and P_y.

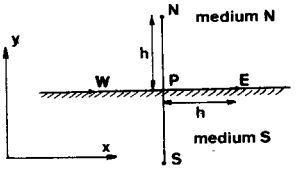

Fig. 2. Horizontal dielectric interface.

The same reasoning when applied to the continuity of $\partial P_x/\partial y$ leads to an iteration relation which is the same as in a homogeneous region with a permittivity ϵ_m. This shows that if the horizontal interfaces were the only ones (this is the case for dielectric slabs), it would be possible to compute P_x alone and, from this, the values of P_y: the vector procedure is not necessary, the problem being actually a scalar one. When both types of dielectric interfaces are present, corresponding equations are found for P_x and P_y, and the two components have to be computed simultaneously. The algebraic system to be solved for the computation of each iterate can be written as follows:

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]\left[\begin{array}{c} P_{x(1,1)} \\ \cdots \\ P_{x(m,n)} \\ \hline P_{y(1,1)} \\ \cdots \\ P_{y(m,n)} \end{array}\right]_n = \left[M\right]\left[\begin{array}{c} P_{x(1,1)} \\ \cdots \\ P_{x(m,n)} \\ \hline P_{y(1,1)} \\ \cdots \\ P_{y(m,n)} \end{array}\right]_{n-1} \tag{25}$$

where the M matrix is a diagonal one, and B (also C) contains nonzero elements only on the lines corresponding to points lying on a vertical (also horizontal) dielectric interface.

The system (25) may be solved by direct matrix inversion, taking advantage of the particular structure of the matrix. The first results were obtained by using this method. However, matrix inversion requires an important memory space. This severely restricts the number of discretization points, hence the quality of the approximation. As an example, an IBM 360/44 computer does not permit a 20 × 40 mesh to be used without using magnetic tapes or disks, which requires a nonnegligible time. To avoid this drawback, iterative methods were developed for solving the system.

Since each iteration of the variation–iteration procedure (contrary to alternate methods) only requires the computation of a function from another one, there is no need for the introduction of an estimate for the unknown eigenvalue: the diagonal dominance of (25) is then obtained (except for points lying on the interfaces but this does not destroy the convergence) and methods as powerful as the successive overrelaxation (SOR) method may be applied. This method has been found to be a reasonable compromise between ease of application and convergence speed. Carré's method [17] is used for evaluating the optimum accelerating factor. The equations for both components being almost uncoupled, it has been found advantageous (although not indispensable) to compute two different accelerating factors.

A special care has been devoted to this evaluation because once the accelerating factor is found for the first iterate, it may be used unchanged for the other iterations, the system matrix remaining unchanged. An original procedure allows the extrapolation of this factor to other mesh sizes (modification of h) or to other points of the dispersion characteristic (modification of β) [13], [14].

The SOR method enables the computation of the successive iterates converging to the eigenvector X_0, from which the electromagnetic fields are computed.

As stated before, the interface conditions coupling P_x and P_y make the problem non-self-adjoint when both types of interfaces are present. The use of (7) and (8) requires the simultaneous solution of the adjoint problem. This can be avoided by noticing that the following property of the scalar products contained in the Rayleigh quotients:

$$(\mathfrak{M} f_n, g_m{}^a) = (\mathfrak{M} f_{n+r}, g_{m-r}{}^a) \tag{26}$$

leads to

$$(\mathfrak{M} f_n, g_n{}^a) = (\mathfrak{M} f_{2n}, g_0{}^a). \tag{27}$$

Since $g_0{}^a$ is an arbitrary trial function of the adjoint problem, it is clear that performing $2n$ iterations on the problem or n iterations in parallel with n iterations on the adjoint problem give the same results. The problem being almost self-adjoint, the last computed iterate f_n is a good estimation for $g_0{}^a$. This leads to the replacement of $g_n{}^a$ in (7) and (8) by f_n but it must be kept in mind that the results obtained after n iterations present the same quality as those obtained with $n/2$ iterations on the problem and $n/2$ on its adjoint (or $n/2$ iterations in the self-adjoint case).

In matrix notation and with the classical scalar product, the generalized Rayleigh quotients are

$$\lambda_0{}^{(n)} = \frac{\iint (P_{x(n-1)}{}^t(M)P_{xn})\,dx\,dy + \iint (P_{y(n-1)}{}^t(M)P_{yn})\,dx\,dy}{\iint (P_{xn}{}^t(M)P_{xn})\,dx\,dy + \iint (P_{yn}{}^t(M)P_{yn})\,dx\,dy} \tag{28}$$

$$\lambda_0{}^{(n+1/2)} = \frac{\iint (P_{xn}{}^t(M)P_{xn})\,dx\,dy + \iint (P_{yn}{}^t(M)P_{yn})\,dx\,dy}{\iint (P_{x(n+1)}{}^t(M)P_{xn})\,dx\,dy + \iint (P_{y(n+1)}{}^t(M)P_{yn})\,dx\,dy} \tag{29}$$

where t indicates the transposition and (M) the matrix form of the operator $\mathfrak{M}$ including the boundary and interface conditions.

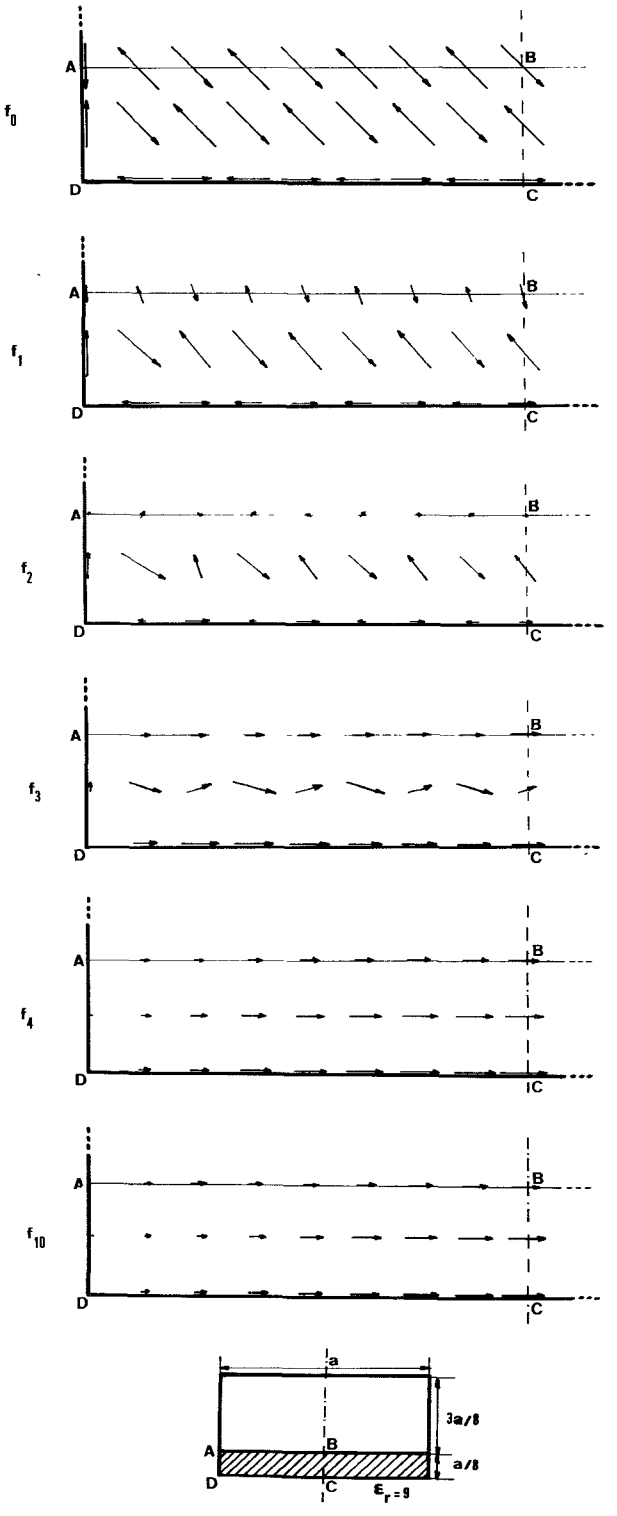

Fig. 3. Transverse magnetic field for successive iterates, f_0 being the trial field (only the *ABCD* section of the guide is represented).

In this non-self-adjoint case, the monotonic convergence of the $\lambda_0^{(a)}$ appears only after some iterations, depending upon the choice of the trial vector.

In order to compute the higher order modes, it is possible to orthogonalize by using the fact that an eigenfunction corresponding to an eigenvalue λ_n is orthogonal to each eigenfunction of the adjoint problem which does not correspond to λ_n*. By using the generalized Rayleigh quotients, we can avoid solving the adjoint problem. It is then possible to use more fundamental orthogonality properties for fields in an inhomogeneous (but isotropic) loaded waveguide:

$$\iint (\bar{E}_{it} \times \bar{H}_{jt}) \cdot \overline{dS} = 0, \qquad \text{if } i \neq j \tag{30}$$

which is easily written in terms of P_x and P_y.

It must be emphasized here that, in order to yield accurate results, the orthogonalization has to be performed upon lower order eigenfunctions obtained with a greater accuracy than previously required. The variational property of the Rayleigh quotient was used indeed for the evaluation of the corresponding eigenvalues. Hence more iterations are required to refine the eigenfunctions if they are to be used for orthogonalization. Moreover, each lower order eigenfunction has to be stored. These two facts limit the use of the method to 3 or 4 higher order modes, depending upon the mesh size.

Fig. 3 shows the efficiency of the variation–iteration method by representing the transverse magnetic field corresponding to successive iterates. Starting from a rather complicated trial field, four iterations are enough to give to the field its final form.

Fig. 4 shows the influence of the mesh used for the calculation of the iterates. Classical techniques may be used as the progressive mesh refining using as initial function the (interpolated) solution of a coarser net. This allows the economy of (time consuming) iterations on a fine net by a supplement of (faster) iterations on a coarse net. Furthermore, an extrapolation based on "Richardson's deferred approach to the limit" leads to a better value of the unknown solution by estimation of the discretization error.

Fig. 5 shows dispersion characteristics of a dielectric loaded waveguide while Figs. 6–8 represent the transverse magnetic fields of some modes of the structure.

IV. Application to a Waveguide Containing a Dielectric Lossy Insert

The variation–iteration method may be used in the complex case [18] provided that the set of eigenfunctions remains a complete one. The method then converges to the eigenfunction corresponding to the eigenvalue presenting the smallest modulus, the convergence rate being given by $|\lambda_1| / |\lambda_0|$. The choice of the eigenvalue, however, has to be reviewed. γ^2 is now a complex quantity and it is no more possible to treat it as a parameter. One has to calculate the complex eigenvalues γ^2 corresponding to given values of k_0^2 (i.e., the frequency) and write (18) under the form

$$(\nabla_t^2 + \epsilon_r k_0^2)\bar{P} = -\gamma^2 \bar{P} \tag{31}$$

where ϵ_r, γ^2, and $\bar{P}$ are complex.

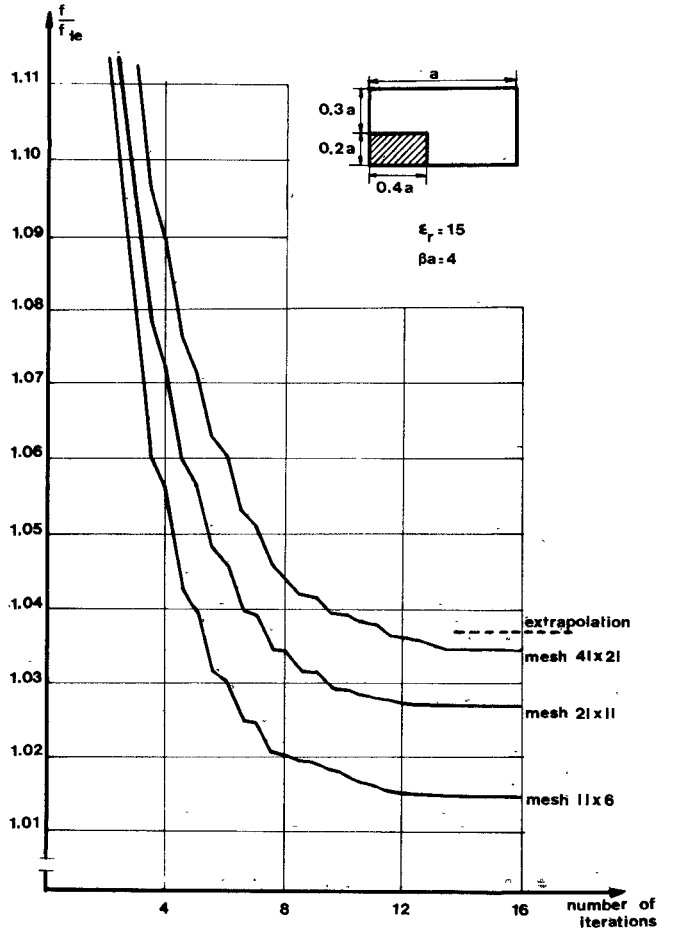

Fig. 4. Influence of the mesh used for the calculation of the iterates (f_{de} is the cutoff frequency of the dominant mode in the empty guide).

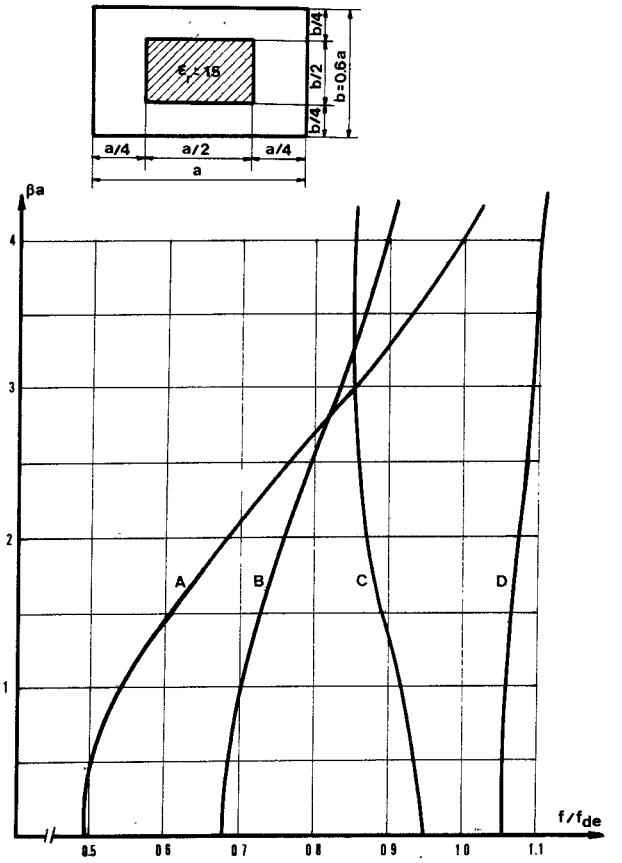

Fig. 5. Dispersion characteristics of a dielectric loaded waveguide (f_{de} is the cutoff frequency of the dominant mode in the empty guide).

Fig. 9 shows the dispersion characteristic of a waveguide with an homogeneous lossless load in the (γ^2, k_0^2) plane. The procedure used in the lossless case corresponds to a search for the successive modes on a vertical line (v), γ^2 being given: the dominant mode (A) is first found. In the lossy case, the search proceeds along an horizontal line (h), k_0^2 being the parameter. The smallest value of $|\gamma^2|$ does not necessarily correspond to the dominant mode (convergence towards B' will occur at the first time). It is necessary to shift the k_0^2 axis to the left, replacing γ^2 by $\gamma^2 + T$, with T a complex constant, so that $|\gamma_0^2 + T| < |\gamma_1^2 + T| < \cdots$. Equation (31) is then replaced by

$$(\nabla_t^2 + \epsilon_r k_0^2 - T)\bar{P} = -(\gamma^2 + T)\bar{P}. \qquad (32)$$

In an homogeneous region, the discretization of (32) gives for the iteration relation

$$(P_N + P_S + P_E + P_W - (4 - k_0^2\epsilon_r h^2 + Th^2)P_P)_n = -h^2(P_P)_{n-1}. \qquad (33)$$

It may be seen that, without shifting the eigenvalues, there is no diagonal dominance and the SOR method does not converge. This shift is thus required (fortunately in the same direction) both by the variation–iteration procedure as well as the SOR method used for calculating

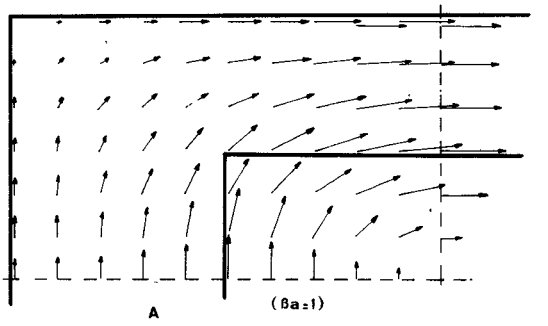

Fig. 6. Transverse magnetic field for the A mode of Fig. 5.

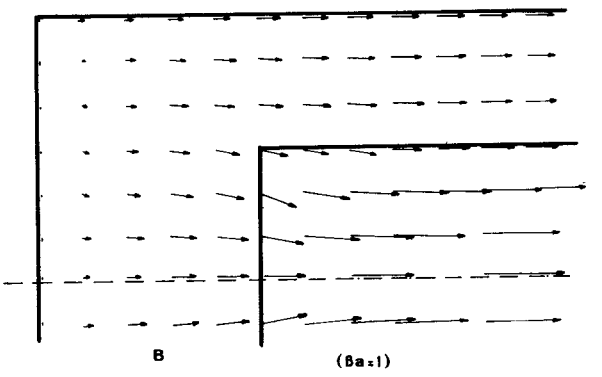

Fig. 7. Transverse magnetic field for the B mode of Fig. 5.

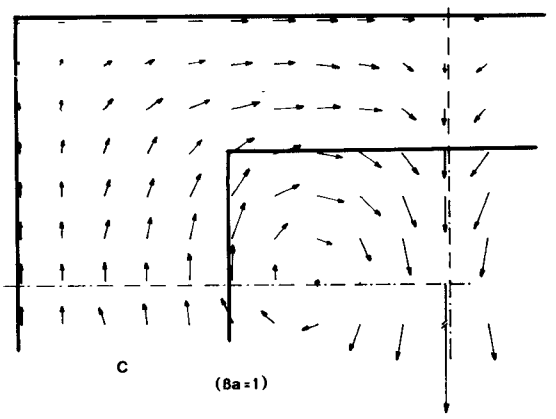

Fig. 8. Transverse magnetic field for the C mode of Fig. 5.

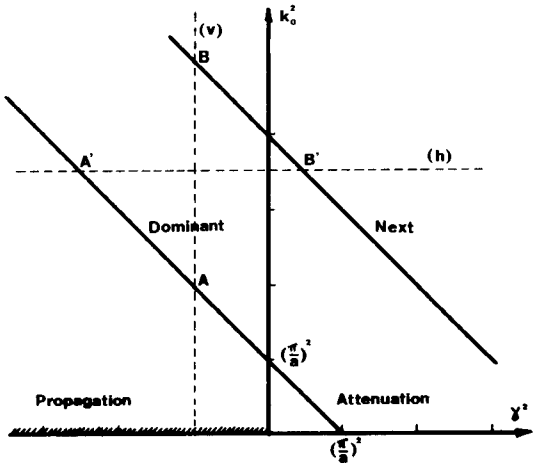

Fig. 9. (γ^2,k_0^2) plane for a waveguide with homogeneous lossless load.

the iterates. T has to be chosen large enough to lead to a converging relaxation procedure but as small as possible to make the iteration procedure as fast as possible (the eigenvalue shift reduces the ratio $|\lambda_1|/|\lambda_0|$ which determines the convergence speed).

Fig. 10 shows that for small values of $\tan\delta$ (<0.01), γ^2 is almost real and it is quite easy to find a good estimate for T. When $\tan\delta$ increases, the search for a reasonable T becomes more difficult.

There are no major difficulties involved in solving the finite-difference system by the SOR procedure with a complex accelerating factor [19]. As mentioned earlier, it was found that the monotonic convergence appears after some iterations as in the lossless case. Hence the extension of the real procedure to the complex vector one is rather straightforward. As in the real case, no preliminary knowledge of the potential is required (except for accelerating the convergence); an eigenvector corresponding to a vector potential aligned with the x axis may be obtained by starting from any trial function, the y component vanishing during the first iterations. In the complex case, the same property exists whatever the "complex degree" of the trial potential—it may be real, imaginary, or complex—the procedure will give it its definite characteristics.

When describing a dispersion characteristic, it is obvious that the use of the solution for a given point as the trial vector for the next point accelerates the procedure.

Fig. 11 gives the dispersion characteristics of waveguides loaded with, respectively, a dielectric slab ($\tan\delta = 0.1$) and a dielectric insert ($\tan\delta = 0.01$). For the insert, two different scales are used to amplify the small variations of α and β.

V. Conclusions

The variation–iteration method has been applied to waveguides loaded with lossy dielectric inserts. Starting from a complex initial trial function, an iteration technique rapidly eliminates the contribution from all eigenfunctions except the one with the lowest eigenvalue contained in the trial function. The complex SOR method is used to solve the partial differential equation giving each iterate. A typical computation needing 6 iterates (this depends strongly upon the quality of the initial function) takes about 50 seconds (including the computation time of the electromagnetic fields) on an IBM 370/155 when a mesh size of about 40×30 is used for the computation of the iterates.

It is the authors' belief that this method makes the exact solution of the complex vector Helmholtz equation available. The required number of iterations never be-

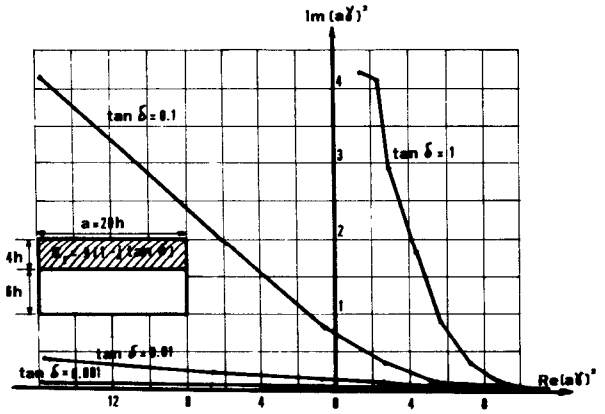

Fig. 10. Complex values of γ^2 in function of $\tan\delta$.

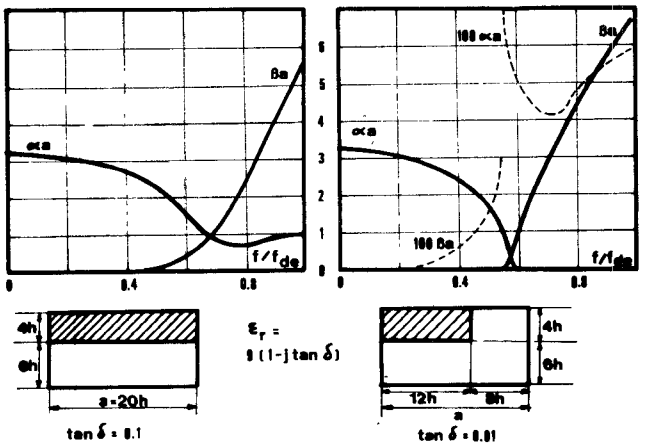

Fig. 11. Dispersion characteristics of loaded waveguides (f_{de} is the cutoff frequency of the dominant mode in the empty guide).

comes prohibitively large because of the high speed of convergence.

It is to be noticed here that the fields are easily calculated and that the concentration of the fields in the dielectric load can be evaluated as a function of the dielectric constant and of the frequency. Hence breakdown, for instance, can be calculated as well as the geometry of a detector to be used in a loaded waveguide. The waveguide impedance concept may be explored. A complete study of the backward modes is presently in progress. The method has also been used to study the problem of waveguides loaded with a strip conductor on a dielectric substrate [14]. The waveguide modes of the structure are easily computed by the described procedure, but the quasi-TEM mode needs a more intricate formulation. The extension of the procedure to anisotropic loads is presently planned.

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References

[1] A. S. Vander Vorst and R. J. M. Govaerts, "Application of a variation–iteration method to inhomogeneously loaded waveguides," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-18, pp. 468–475, Aug. 1970.

[2] A. Lavik and H. G. Unger, "Der Rechteckhohlleiter mit Rechteckigen Stoffeinsatz," *A.E.U.*, vol. 18, pp. 25–34, 1964.

[3] ——, "Rückwärtswellen in Homogenen Wellenleitern," *A.E.U.*, vol. 18, pp. 35–42, 1964.

[4] W. Schlosser and H. G. Unger, "Partially filled waveguides and surface waveguides of rectangular cross-section," in *Advances in Microwaves*, vol. 1, L. Young, Ed. New York: Academic, 1966.

[5] W. Baier, "Waves and evanescent fields in rectangular waveguides filled with a transversely inhomogeneous dielectric," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-18, pp. 696–705, Oct. 1970.

[6] Z. J. Csendes and P. Silvester, "Numerical solution of dielectric loaded waveguides: I—Finite-element analysis," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-18, pp. 1124–1131, Dec. 1970.

[7] A. S. Vander Vorst, A. A. Laloux, and R. J. M. Govaerts, "A computer optimization of the Rayleigh-Ritz method," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-17, pp. 454–460, Aug. 1969.

[8] Z. J. Csendes and P. Silvester, "Numerical solution of dielectric loaded waveguides: II—Modal approximation technique," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-19, pp. 504–509, June 1971.

[9] H. Wielandt, "Das Iterationsverfahren in der Flatterrechnung," *A.V.A. Bericht*, B44/J/21, 1944.

[10] ——, "Konvergenzbeweis für das Iterationsverfahren," *A.V.A. Bericht*, B44/J/38, 1944.

[11] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*. New York: McGraw-Hill, 1953, ch. 9.

[12] R. Lattès, *Quelques Méthodes de Résolution des Problèmes aux Limites de la Physique Mathématique*. New York: Gordon and Breach, 1967.

[13] R. Govaerts, "Propagation in rectangular waveguides partially loaded with dielectric inserts." Ph.D. dissertation, Catholic Univ. Louvain, Louvain, Belgium, 1971.

[14] A. Laloux, "Etude numérique de guides d'ondes à charge diélectrique inhomogène," Ph.D. dissertation, Catholic Univ. Louvain, Louvain, Belgium, 1973.

[15] A. Vander Vorst and R. Govaerts, "Rectangular waveguides partially filled with dielectric inserts," in *Proc. 4th Colloq. Microwave Communications* (Budapest, Hungary), pp. ET 28/1–9, 1970.

[16] C. Lanczos, *Linear Differential Operators*. London, England: Van Nostrand, 1961.

[17] B. A. Carré, "The determination of the optimum accelerating factor for successive overrelaxation," *Comput. J.*, vol. 4, pp. 73–78, 1961.

[18] H. Wielandt, "Beiträge zur Matematischen Behandlung Komplexer Eigenwertprobleme," *A.V.A. Bericht*, 1944.

[19] D. M. Young, *Iterative Solutions of Large Linear Systems*. New York: Academic, 1971.